Lie Algebras All of Whose Proper Subalgebras are Nilpotent

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ABSTRACT

Nonnilpotent Lie algebras all of whose proper subalgebras are nilpotent are studied. A fairly complete description is given of the nonperfect algebras in this class over a wide range of fields (including all perfect fields).

1. INTRODUCTION

All algebras considered will be finite-dimensional over a field F. Let L be a nonnilpotent Lie algebra all of whose proper subalgebras are nilpotent. If F is algebraically closed, it is easy to determine the structure of L. For, since L is not nilpotent, there is an element $x \in L$ such that $\operatorname{ad} x$ has a nonzero eigenvalue λ. Then, if y is a corresponding eigenvector, we have that $xy = \lambda y$ and the subalgebra spanned by x and y is not nilpotent. Thus, L must be the two-dimensional nonabelian Lie algebra.

Of course, one wouldn't expect anything so restrictive over non-algebraically-closed fields, and it is the purpose of this paper to study the possible structures for L over more general fields F. This problem has also been studied by Stitzinger [3] and by Gein and Kuznecov [1].

The basic structure theorem for nonperfect algebras is obtained in Sec. 2, and then a more detailed study of the structure is carried out in Secs. 3 and 4. Finally, as an illustration of the results, the algebras of the title are classified completely when F is the real field. The methods of proof are

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entirely elementary, and are applications of linear algebra and polynomial theory.

If $x_1,\dots,x_n \in L$, we shall denote the subspace of L and the subalgebra of L generated by $x_1,\dots,x_n$ by $((x_1,\dots,x_n))$ and $\langle x_1,\dots,x_n\rangle$ respectively. The product of $x, y \in L$ in L will be denoted by $[x,y]$. Direct sums of the vector space structure of L will be denoted by $\dot{+}$.

2. THE BASIC STRUCTURE THEOREM

The following result, although not stated in quite the same way, was obtained by Stitzinger in [3]; we suspect that something very similar was also obtained by Gein and Kuznecov in [1]. Our proof is different, however, and is an application of linear algebra, and so we include it for completeness.

THEOREM 2.1. *Let L be a nonperfect Lie algebra (i.e. $L \neq L^2$) which is not nilpotent. Then the following are equivalent*:

(i) *all proper subalgebras of L are nilpotent, and*

(ii) *if N is the nilradical of L, then $L = N \dot{+} ((x))$, $N = N^2 \dot{+} ((e_1,\dots,e_r))$, and*

$$[e_1,x]=e_2,\dots,\qquad [e_{r-1},x]=e_r,\qquad [e_r,x]=c_0e_1+\cdots+c_{r-1}e_r,$$

where $c_0 \neq 0$, the polynomial $p(Y) = Y^r - c_{r-1}Y^{r-1} - \cdots - c_1Y - c_0$ is irreducible in $F[Y]$, and $\operatorname{ad}x|_{N^2}$ *is nilpotent.*

Proof.

(a) Suppose first that (i) holds. Since L is not perfect, it has an ideal of codimension 1 in L, and this must be N. Thus, $L = N \dot{+} ((x))$, for some $x \in L$. Also, since L is not nilpotent, $\operatorname{ad}x|_N$ is not nilpotent. This implies that the minimum polynomial of $\operatorname{ad}x|_N$ has an irreducible factor $p(Y) = Y^r - c_{r-1}Y^{r-1} - \cdots - c_1Y - c_0$ with $c_0 \neq 0$ $(r \geqslant 1)$. Pick a basis $e_1,\dots,e_n$ for N with respect to which $\operatorname{ad}x|_N$ is represented by a matrix in rational canonical form. One of the companion matrices will correspond to $p(Y)$, so we can assume that $[e_1,x]=e_2,\dots,$ $[e_{r-1},x]=e_r$, $[e_r,x]=c_0e_1+\cdots+c_{r-1}e_r$. Now $\langle e_1,\dots,e_r,x\rangle$ is not nilpotent and hence must be L.

Clearly $N = N^2 + U$ where $U = ((e_1,\dots,e_r))$. Furthermore, $\operatorname{ad}x|_U$ has $p(Y)$ as its minimum polynomial and as its characteristic polynomial. Because $p(Y)$ is irreducible, the only $\operatorname{ad}x$-stable subspaces of U are 0 and U. But $U \cap N^2$ is clearly $\operatorname{ad}x$-stable and $U \not\subseteq N^2$, so $U \cap N^2 = 0$. Finally, $N^2 + ((x))$ is a proper subalgebra of L, so $\operatorname{ad}x|_{N^2}$ is nilpotent.

(b) Suppose now that (ii) holds. We need only prove that maximal subalgebras are nilpotent, so let M be a maximal subalgebra of L with $N \not\subset M$. Then $L = N + M$ and $N^2 \subset M$ (see, for instance, [4, Theorem 6.5]). It follows that $N \cap M$ is an ideal of L and hence that $N \cap M = N^2$ (since N/N^2 has no proper $\operatorname{ad} x$-stable subspace). Clearly then $M = N^2 + ((z))$ where $z \notin N$. But $z = n + \lambda x$, where $n \in N^2$, $\lambda \in F$, so that $\operatorname{ad} z|_{N^2}$ is nilpotent, and thus M is nilpotent. ■

In contrast with the case where F is algebraically closed, we have the following.

COROLLARY 2.2. *There are nonnilpotent solvable Lie algebras of arbitrary dimension over the rational field, and over any finite field, having all proper subalgebras nilpotent.*

Proof. Simply note that over the stated fields there are irreducible polynomials of degree n for every positive integer n. ■

It is easy to solve the isomorphism problem for the algebras described in Theorem 2.1 which have abelian nilradical (and we shall see later that "most" of them do have abelian nilradical).

THEOREM 2.3. *Let L, L' both have the structure described in Theorem 2.1 (use the same notation for L, and put a prime on everything for L'), and suppose that N, N' are both abelian. Then a necessary and sufficient condition for L to be isomorphic to L' is that there is an element $\alpha \in F$ such that $\alpha c_{r-1} = c'_{r-1}$, $\alpha^2 c_{r-2} = c'_{r-2}, \ldots, \alpha^r c_0 = c'_0$.*

Proof. For necessity simply note that if $x' = \alpha x + n$, then $\operatorname{ad} x|_N$ has characteristic polynomials $Y^r - c_{r-1}Y^{r-1} - \cdots - c_1 Y - c_0$ and $Y^r - (c'_{r-1}/\alpha)Y^{r-1} - \cdots - (c'_1/\alpha^{r-1})Y - (c'_0/\alpha^r)$ relative to bases $\{e_1, \ldots, e_r\}$ and $\{e'_1, \ldots, e'_r\}$ respectively.

For sufficiency, put $x' = \alpha x$, $e'_1 = e_1$, $e'_2 = \alpha e_2, \ldots, e'_r = \alpha^{r-1} e_r$. ■

3. THE STRUCTURE OF THE NILRADICAL

The description of L given in Theorem 2.1 is incomplete in at least two ways. We do not know which nilpotent Lie algebras N can occur as the nilradical of such an algebra, and we do not know exactly how $\operatorname{ad} x$ acts on N^2 (unless, of course, $N^2 = 0$). We shall tackle the first of these problems in this section, and the second in the next section. Throughout this section L, N, and the other notation will be as in Theorem 2.1.

The following was proved by Stitzinger in [3].

Theorem 3.1 (Stitzinger). $N^3 = 0$.

It is this result that we want to improve on.

Theorem 3.2.

(i) *If N has an odd number of generators (in the notation of Theorem 2.1, $e_1, \ldots, e_r$ are the generators of N), then N is abelian.*

(ii) *If N has an even number of generators and $c_{2i+1} \neq 0$ for any $0 \leqslant i \leqslant (r-2)/2$, then N is abelian.*

Proof. (i):

(a) Suppose first that $\operatorname{ad} x|_{N^2} = 0$. Put $r = 2n+1$. Then

$$\begin{aligned} 0 &= [[e_i, e_j], x] = -[[e_j, x], e_i] - [[x, e_i], e_j] \\ &= [e_i, e_{j+1}] + [e_{i+1}, e_j] \qquad \text{for} \quad i < j \leqslant r-1, \end{aligned} \tag{1}$$

and

$$\begin{aligned} 0 &= [[e_i, e_r], x] = -[[e_r, x], e_i] - [[x, e_i], e_r] \\ &= c_0[e_i, e_1] + c_1[e_i, e_2] + \cdots + c_{r-1}[e_i, e_r] + [e_{i+1}, e_r] \qquad \text{for} \quad i < r. \end{aligned} \tag{2}$$

Now (1) implies that

$$[e_i, e_j] = 0 \qquad \text{for} \quad i + j \equiv 0 \pmod 2, \quad 1 \leqslant i < j \leqslant r,$$

and (3)

$$[e_i, e_j] = \begin{cases} (-1)^{i-1}[e_1, e_{j+i-1}] & \text{for} \quad 1 \leqslant i < j \leqslant r+1-i \\ (-1)^{r-i}[e_{i-r+j}, e_r] & \text{for} \quad \max\{i, r+1-i\} \leqslant j \leqslant r. \end{cases}$$

Using (3), we can write the system of equations (2) in terms of $[e_1, e_2]$, $[e_1, e_4], \ldots, [e_1, e_{2n}]; [e_2, e_r], [e_4, e_r], \ldots, [e_{2n}, e_r]$ alone; thus (2) becomes

$$\begin{array}{ll} (i=1) & c_1[e_1,e_2] + c_3[e_1,e_4] + \cdots + c_{2n-1}[e_1,e_{2n}] + [e_2,e_{2n+1}] = 0, \\ (i=2) & -c_0[e_1,e_2] - c_2[e_1,e_4] - \cdots - c_{2n-2}[e_1,e_{2n}] + c_{2n}[e_2,e_{2n+1}] = 0, \\ (i=3) & c_1[e_1,e_4] + \cdots + c_{2n-3}[e_1,e_{2n}] - c_{2n-1}[e_2,e_{2n+1}] \\ & \qquad + [e_4,e_{2n+1}] = 0, \end{array}$$

and so on. This is a system of $2n$ linear homogeneous equations in the $2n$ unknowns $[e_1,e_2],[e_1,e_4],\ldots,[e_1,e_{2n}];[e_2,e_r],[e_4,e_r],\ldots,[e_{2n},e_r]$ with matrix of coefficients

$$C=\begin{bmatrix} c_1 & c_3 & \cdots & c_{2i+1} & \cdots & c_{2n-1} & 1 & 0 & 0 & \cdots & 0 \\ -c_0 & -c_2 & \cdots & -c_{2i} & \cdots & -c_{2n-2} & c_{2n} & 0 & 0 & \cdots & 0 \\ 0 & c_1 & \cdots & c_{2i-1} & \cdots & c_{2n-3} & -c_{2n-1} & 1 & 0 & \cdots & 0 \\ 0 & -c_0 & \cdots & -c_{2i-1} & \cdots & -c_{2n-4} & c_{2n-2} & c_{2n} & 0 & \cdots & 0 \\ \hdotsfor{11} \\ 0 & \cdots & \cdots & \cdots & 0 & c_1 & -c_3 & -c_5 & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & -c_0 & c_2 & c_4 & \cdots & \cdots & c_{2n} \end{bmatrix}.$$

Multiplying each of the first n columns by -1 and performing row and column interchanges, C can be transformed into the matrix

$$D=\begin{bmatrix} c_{2n} & c_{2n-2} & \cdots & \cdots & c_0 & 0 & 0 & \cdots & 0 \\ 0 & c_{2n} & \cdots & \cdots & c_2 & c_0 & 0 & \cdots & 0 \\ \hdotsfor{9} \\ 0 & \cdots & 0 & c_{2n} & c_{2n-2} & \cdots & \cdots & \cdots & c_0 \\ 1 & -c_{2n-1} & \cdots & \cdots & -c_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & -c_3 & -c_1 & 0 & \cdots & 0 \\ \hdotsfor{9} \\ 0 & \cdots & 0 & 1 & -c_{2n-1} & \cdots & \cdots & \cdots & -c_1 \end{bmatrix}.$$

Now suppose that N is nonabelian. Then we must have that $\det C=0$, which implies that $\det D=0$. But $\det D$ is Sylvester's determinant form of the resolvent[1] for the polynomials

$$f(Y)=c_{2n}Y^n+c_{2n-2}Y^{n-1}+\cdots+c_0$$

and

$$g(Y)=Y^n-c_{2n-1}Y^{n-1}-\cdots-c_1.$$

If $\det D=0$ it follows that $f(Y)$ and $g(Y)$ have a common factor $h(Y)$ of degree $\geqslant 1$. But this means that

$$\begin{aligned} p(Y) &= Y^{2n+1}-c_{2n}Y^{2n}-\cdots-c_1Y-c_0 \\ &= Y\left(Y^{2n}-c_{2n-1}Y^{2n-2}-\cdots-c_1\right)-\left(c_{2n}Y^{2n}+c_{2n-2}Y^{2n-2}+\cdots+c_0\right) \\ &= Yf(Y^2)-g(Y^2) \end{aligned}$$

[1]Also known as the resultant (see, for example, Van der Waerden, Modern Algebra, Vol. I, p. 84).

and so $h(Y^2)$ is a factor of $p(Y)$. But $p(Y)$ is irreducible and $1 \leqslant$ degree $h(Y^2) \leqslant 2n <$ degree $p(Y)$, which is impossible. Hence N must be abelian.

(b) So suppose now that $\operatorname{ad} x|_{N^2} \neq 0$. Since $N^3 = 0$, it is easy to see that $[N^2, x]$ (the image of N^2 under $\operatorname{ad} x$) is an ideal of L. By case (a), $N/[N^2, x]$ is abelian, and so $N^2 \subset [N^2, x]$. But this means that $N^2 \subset N^2(\operatorname{ad} x)^n$ for all positive integers n. The fact that $\operatorname{ad} x|_{N^2}$ is nilpotent thus implies that N is abelian.

(ii): We need only prove the result under the assumption that $\operatorname{ad} x|_{N^2} = 0$; the general result will then follow as in (i) (b), except that this time no sign changes are necessary. Put $r = 2n$. Equations (2) and (3) remain the same as in (i), and we can write the system of equations (2) in terms of $[e_1, e_2], [e_1, e_4], \ldots, [e_1, e_{2n}]; [e_3, e_{2n}], [e_5, e_{2n}], \ldots, [e_{2n-1}, e_{2n}]$ alone; thus (2) becomes

$$\begin{array}{lrr}
(i=1) & c_1[e_1,e_2] + c_3[e_1,e_4] + \cdots + c_{2n-1}[e_1,e_{2n}] & = 0, \\
(i=2) & -c_0[e_1,e_2] - c_2[e_1,e_4] - \cdots - c_{2n-2}[e_1,e_{2n}] + \qquad [e_3,e_{2n}] & = 0, \\
(i=3) & c_1[e_1,e_4] + \cdots + c_{2n-3}[e_1,e_{2n}] + c_{2n-1}[e_3,e_{2n}] & = 0,
\end{array}$$

and so on. This time

$$C = \begin{bmatrix}
c_1 & c_3 & \cdots & c_{2i+1} & \cdots & c_{2n-1} & 0 & 0 & 0 & \cdots & 0 \\
-c_0 & -c_2 & \cdots & -c_{2i} & \cdots & -c_{2n-2} & 1 & 0 & 0 & \cdots & 0 \\
0 & c_1 & \cdots & c_{2i-1} & \cdots & c_{2n-3} & c_{2n-1} & 0 & 0 & \cdots & 0 \\
0 & -c_0 & \cdots & -c_{2i-2} & \cdots & -c_{2n-4} & -c_{2n-2} & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & 0 & c_1 & -c_3 & -c_5 & \cdots & \cdots & -c_{2n-1}
\end{bmatrix},$$

$$D = \begin{bmatrix}
1 & -c_{2n-2} & \cdots & \cdots & \cdots & -c_0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & \cdots & \cdots & -c_2 & -c_0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & -c_{2n-2} & -c_{2n-4} & \cdots & \cdots & \cdots & -c_0 \\
c_{2n-1} & c_{2n-3} & \cdots & \cdots & c_1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & c_{2n-1} & \cdots & \cdots & c_3 & c_1 & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & c_{2n-1} & c_{2n-3} & \cdots & \cdots & \cdots & c_1
\end{bmatrix},$$

and $\det D$ is the resolvent for

$$f(Y) = Y^n - c_{2n-2}Y^{n-1} - \cdots - c_0$$

and

$$g(Y) = c_{2n-1}Y^{n-1} + c_{2n-3}Y^{n-2} + \cdots + c_1.$$

As before, $\det C=0$ implies that $f(Y)$ and $g(Y)$ have a common factor $h(Y)$ of degree $\geqslant 1$. This means that

$$\begin{aligned}p(Y) &= Y^{2n}-c_{2n-1}Y^{2n-1}-\cdots-c_1Y-c_0\\ &=\left(Y^{2n}-c_{2n-2}Y^{2n-2}-\cdots-c_0\right)\\ &\quad -Y\left(c_{2n-1}Y^{2n-2}+c_{2n-3}Y^{2n-4}+\cdots+c_1\right)\\ &=f(Y^2)-Yg(Y^2),\end{aligned}$$

and so $h(Y^2)$ is a factor of $p(Y)$. Since $p(Y)$ is irreducible and has degree $\geqslant 1$, we must have $h(Y^2)=p(Y)$. It follows that $p(Y)=f(Y^2)$ and $g(Y)=0$. If $g(Y)\neq 0$, we therefore have our contradiction as before. ■

4. THE ACTION OF ad$x$

The previous section leaves open the question of how ad$x$ acts on N^2 when $N^2\neq 0$. This is answered in a large number of cases by the following result. In the light of Theorem 3.2 we may assume that $c_{2i+1}=0$ for all $0\leqslant i\leqslant n-1$.

THEOREM 4.1. *Let L,N be as in Theorem* 2.1, *and suppose that* $\dim(N/N^2)=r=2n$. *Then either*

(i) $\operatorname{ad}x|_{N^2}=0$, *or*

(ii) *F has characteristic p (>0), p divides n, and $c_{2i}=0$ for all $0<i\leqslant n-1$ for which p does not divide i.*

Proof.

(a) Suppose first that $(\operatorname{ad}x)^2|_{N^2}=0$. Then

$$0=[[[e_i,e_j],x],x]=[[e_i,e_{j+1}],x]+[[e_{i+1},e_j],x]$$

$$=\begin{cases}[e_i,e_{j+2}]+2[e_{i+1},e_{j+1}]+[e_{i+2},e_j] & \text{for}\quad i<j\leqslant r-2, \quad (1)\\ c_0[e_i,e_1]+c_2[e_i,e_3]+\cdots+c_{r-2}[e_i,e_{r-1}] & \\ \quad +2[e_{i+1},e_r]+[e_{i+2},e_{r-1}] & \text{for}\quad i<r-1, \quad (2)\end{cases}$$

$$
\begin{aligned}
0&=[[[e_i,e_r],x],x]\\
&=c_0[[e_i,e_1],x]+\cdots+c_{r-2}[[e_i,e_{r-1}],x]+[[e_{i+1},e_r],x]\\
&=c_0([e_i,e_2]+[e_{i+1},e_1])+\cdots+c_{r-2}([e_i,e_r]+[e_{i+1},e_{r-1}])\\
&\quad+c_0[e_{i+1},e_1]+\cdots+c_{r-2}[e_{i+1},e_{r-1}]+[e_{i+2},e_r]\\
&=c_0([e_i,e_2]+2[e_{i+1},e_1])+\cdots+c_{r-2}([e_i,e_r]+2[e_{i+1},e_{r-1}])\\
&\quad+[e_{i+2},e_r] \qquad \text{for} \quad i<r-1. \qquad (3)
\end{aligned}
$$

Hence

$$[[e_i,e_j],x]=0 \qquad \text{for} \quad i+j\equiv 0 \pmod 2). \qquad (4)$$

We shall need the following lemma.

LEMMA. $[e_i, e_j] = (-1)^{m+1}m[e_{i+m-1}, e_{j-m+1}]$ *if* $j-i=2m$ *and* $2+i\leqslant j\leqslant r$.

Proof. We use induction on m, the result being clear if $m=1$. Suppose that it holds for $m\leqslant k$, and let $j-i=2(k+1)$. Then

$$
\begin{aligned}
[e_i,e_j]&=-2[e_{i+1},e_{j-1}]-[e_{i+2},e_{j-2}]\\
&=(-1)^{k+1}(-2)k[e_{i+k},e_{j-k}]-(-1)^k(k-1)[e_{i+k},e_{j-k}]\\
&\qquad[\text{since} \quad (j-1)-(i+1)=2k,\ (j-2)-(i+2)=2(k-1)]\\
&=(-1)^{k+2}(k+1)[e_{i+k},e_{j-k}]. \qquad \blacksquare
\end{aligned}
$$

Using this lemma we can write the equations (2) and (3) in terms of $[e_1,e_3]$, $[e_2,e_4]$, $[e_3,e_5],\ldots,[e_{r-2},e_r]$ alone. We consider only the equations (2) with $i\equiv 1 \pmod 2$ and the equations (3) with $i\equiv 0 \pmod 2$. These become

(2):

$$
\begin{aligned}
(i=1)\qquad & c_2[e_1,e_3]-2c_4[e_2,e_4]+3c_6[e_3,e_5]+\cdots+(-1)^n(n-1)\\
&\times c_{r-2}[e_{n-1},e_{n+1}]+(-1)^n n[e_n,e_{n+2}]=0,
\end{aligned}
$$

$$(i=3) \qquad -c_0[e_1,e_3]+c_4[e_3,e_5]-2c_6[e_4,e_6]+\cdots+(-1)^{n-1}(n-2) \times c_{r-2}[e_n,e_{n-1}]+(-1)^{n-1}(n-1)[e_{n+1},e_{n+3}]=0,$$

and so on (up to $i=r-3$);

(3):

$$(i=2) \qquad -2c_0[e_1,e_3]+c_2[e_2,e_4]-c_6[e_4,e_6]+2c_8[e_5,e_7]+\cdots +(-1)^{n+1}(n-3)c_{r-2}[e_n,e_{n+2}]+(-1)^{n+1}(n-2)[e_{n+1},e_{n+3}]=0,$$

$$(i=4) \qquad 3c_0[e_2,e_4]-2c_2[e_3,e_5]+c_4[e_4,e_6]-c_8[e_6,e_8]+\cdots+(-1)^n \times(n-4)c_{r-2}[e_{n+1},e_{n+2}]+(-1)^n(n-3)[e_{n+2},e_{n+4}]=0,$$

and so on (up to $i=r-2$).

This is a system of $2n-2$ linear homogeneous equations in the $2n-2$ unknowns $[e_1,e_3],[e_2,e_4],\ldots,[e_{2n-2},e_{2n}]$ with matrix of coefficients

$$C=\left[\begin{array}{ccccccccccc}
c_2 & -2c_4 & 3c_6 & -4c_8 & \cdots & (-1)^n(n-1)c_{r-2} & (-1)^n n & 0 & \cdots & \cdots & 0 \\
-c_0 & 0 & c_4 & -2c_6 & \cdots & \cdots & (-1)^{n-1}(n-2)c_{r-2} & (-1)^{n-1}(n-1) & 0 & \cdots & 0 \\
0 & 2c_0 & -c_2 & 0 & c_6 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & -3c_0 & 2c_2 & -c_4 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hdotsfor{11} \\
-2c_0 & c_2 & 0 & -c_6 & 2c_8 & \cdots & (-1)^{n-1}(n-3)c_{r-2} & (-1)^{n-1}(n-2) & 0 & \cdots & 0 \\
0 & 3c_0 & -2c_2 & c_4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hdotsfor{11}
\end{array}\right].$$

For $i=2,\ldots,n-1$ subtract row i from row $i+n-2$; then for $i=2,\ldots,n-1$ subtract row $i+n-2$ from row i. This produces the matrix

$$D=\left[\begin{array}{ccccccccccc}
c_2 & -2c_4 & 3c_6 & \cdots & (-1)^n(n-1)c_{r-2} & (-1)^n n & 0 & \cdots & \cdots & \cdots & 0 \\
0 & -c_2 & 2c_4 & \cdots & (-1)^n(n-2)c_{r-4} & (-1)^{n+1}(n-1)c_{r-2} & (-1)^{n+1}n & 0 & \cdots & \cdots & 0 \\
\hdotsfor{11} \\
-c_0 & c_2 & -c_4 & \cdots & \cdots & (-1)^n c_{r-2} & (-1)^n & 0 & \cdots & \cdots & 0 \\
0 & c_0 & -c_2 & \cdots & \cdots & \cdots & (-1)^n c_{r-2} & (-1)^n & 0 & \cdots & 0 \\
\hdotsfor{11}
\end{array}\right].$$

Multiplying alternate rows by -1 and performing row and column interchanges, we obtain

$$E=\left[\begin{array}{ccccccccc}
n & (n-1)c_{r-2} & \cdots & (-1)^{n-1}2c_4 & (-1)^n c_2 & 0 & \cdots & \cdots & 0 \\
0 & n & \cdots & (-1)^{n-2}3c_6 & (-1)^{n-1}2c_4 & (-1)^n c_2 & 0 & \cdots & 0 \\
\hdotsfor{9} \\
1 & c_{r-2} & \cdots & (-1)^{n-1}c_4 & (-1)^n c_2 & (-1)^{n+1}c_0 & 0 & \cdots & 0 \\
0 & 1 & c_{r-2} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hdotsfor{9}
\end{array}\right].$$

Suppose that $\det E = 0$. Then the polynomials

$$f(Y) = nY^{n-1} + (n-1)c_{r-2}Y^{n-2} + \cdots + (-1)^{n-1}2c_4Y + (-1)^n c_2$$

and

$$g(Y) = Y^n + c_{r-2}Y^{n-1} - c_{r-4}Y^{n-2} + \cdots + (-1)^n c_2 Y + (-1)^{n+1}c_0$$

have a common factor $h(Y)$ of degree $\geqslant 1$. But $p(Y) = (-1)^n g(-Y^2)$, so, by the irreducibility of $p(Y)$, we must have $h(-Y^2) = p(Y)$. Now $f(Y) \neq 0$ implies that $\operatorname{degree} h(-Y^2) \leqslant \operatorname{degree} f(-Y^2) \leqslant 2n-2$, a contradiction. It follows that if $\det E = 0$ then $f(Y) = 0$, and case (ii) holds.

So suppose that $\det E \neq 0$. Then $\det C \neq 0$, which means that $[e_i, e_{i+2}] = 0$ for $1 \leqslant i \leqslant r-2$. But then, by the lemma,

$$[e_i, e_j] = 0 \qquad \text{for all} \quad i < j \leqslant r \ \text{with}\ i + j \equiv 0 \pmod 2.$$

Furthermore, if $i + j \equiv 1 \pmod 2$,

$$[[e_i, e_j], x] = [e_i, e_{j+1}] + [e_{i+1}, e_j] = 0 \qquad \text{for} \quad i < j \leqslant r-1;$$

also, if $i + r \equiv 1 \pmod 2$, then $i \equiv 1 \pmod 2$, so

$$[[e_i, e_r], x] = c_0[e_i, e_1] + c_2[e_i, e_3] + \cdots + c_{r-2}[e_i, e_{r-1}] + [e_{i+1}, e_r] = 0.$$

Hence $[[e_i, e_j], x] = 0$ for all $1 \leqslant i < j \leqslant r$ and case (i) holds.

(b) The general case now follows as in Theorem 3.2 by factoring out the ideal $[[N^2, x], x]$. ■

Corollary 4.2. *Over any perfect field*, $\operatorname{ad} x|_{N^2} = 0$.

The above corollary doesn't hold over all fields. For, let $F = Z_2(t)$, the field of quotients of the ring of polynomials in a single indeterminate over the field of two elements, and let L have basis $\{e_1, e_2, e_3, e_4, e_5, e_6, x\}$ with

$$[e_1, x] = e_2, \quad [e_2, x] = e_3, \quad [e_3, x] = e_4, \quad [e_4, x] = te_1,$$

$$[e_1, e_4] = [e_2, e_3] = e_5, \quad [e_2, e_4] = e_6, \quad [e_5, x] = e_6$$

(all other products being zero). It is easy to check that L is a Lie algebra and that the polynomial $p(Y) = Y^4 - t$ is irreducible in $Z_2(t)[Y]$. However, N^2 is spanned by e_5 and e_6, and $\operatorname{ad} x|_{N^2} \neq 0$.

When $\operatorname{ad}x|_{N^2}=0$, Eqs. (2) and (3) of Theorem 3.2 must hold, and this places restrictions on the possible dimensions for N^2.

THEOREM 4.3. *Let L, N be as in Theorem* 2.1 *and suppose that* $\dim(N/N^2)=r$, *where either r is odd, or else r is even and case* (i) *of Theorem* 4.1 *holds. Then*

(i) $\dim N \leqslant 3r/2$;

(ii) *if* $\dim N = 6n-3$, $6n-1$, *or* $6n+1$, *then* $\dim N^2 \in \{0, 2i-1 : 1 \leqslant i \leqslant n\}$ $(n \geqslant 1)$;

(iii) *if* $\dim N = 6n$, $6n+2$, *or* $6n+4$, *then* $\dim N^2 \in \{0, 2i : 0 \leqslant i \leqslant n\}$ $(n \geqslant 0)$.

Proof. (i): Since $[N^2, x]=0$, Eqs. (2) and (3) of Theorem 3.2 hold. The result is clear if r is odd, and if r is even then N^2 is spanned by $[e_1, e_2]$ $[e_1, e_4], \dots, [e_1, e_r]$. The result follows.

(ii): We have that $\dim N$ is odd, so either $\dim N^2=0$ or $\dim N^2$ is odd. Suppose that $\dim N^2 > 2n-1$. Then

$$\dim \frac{N}{N^2} \leqslant \begin{Bmatrix} 4n-4 \\ 4n-2 \\ 4n \end{Bmatrix} \text{ and } \dim N \leqslant \begin{Bmatrix} 6n-6 \\ 6n-3 \\ 6n \end{Bmatrix} \quad \text{if} \quad \dim N = \begin{Bmatrix} 6n-3 \\ 6n-1 \\ 6n+1 \end{Bmatrix},$$

a contradiction.

(iii): This follows similarly. ∎

5. THE REAL FIELD

We can now classify all the algebras described in the title for the case where F is the real field.

THEOREM 5.1. *Let L be a real Lie algebra. Then all proper subalgebras of L are nilpotent if and only if one of the following holds. (Only nonzero products will be given.)*

(i) *L is nilpotent*;

(ii) *L is* 2-*dimensional with basis* $\{e_1, x\}$ *and multiplication* $[e_1, x]=x$;

(iii) *L is the* 3-*dimensional nonsplit simple Lie algebra*;

(iv) *L is* 3-*dimensional with basis* $\{f_1, f_2, x\}$ *and multiplication*

$$[f_1, x]=f_2, \qquad [f_2, x]=-f_1+c_1 f_2, \quad \text{where } 0 \leqslant c_1 < 2;$$

(v) *L is 4-dimensional with basis* $\{g_1, g_2, g_3, x\}$ *and multiplication*

$$[g_1, g_2] = g_3, \qquad [g_1, x] = g_2, \qquad [g_2, x] = -g_1.$$

No two of the algebras described above (including different members of the family described in (iv)) *are isomorphic.*

Proof.

(a) Suppose first that L is solvable but not nilpotent, so that L has the structure described in Theorem 2.1. Since irreducible polynomials over the real field are linear or quadratic, we have that $r=1$ or 2.

If $r=1$, then N is 1-dimensional with basis $\{e_1\}$, and $[e_1, x] = c_0 e_1$ where $c_0 \neq 0$. Replacing x by $(1/c_0)x$, we see that L has the structure given in (ii).

So we can now suppose that $r=2$. Then $N = N^2 \dotplus U$, where U has basis $\{e_1, e_2\}$, $[e_1, x] = e_2$, $[e_2, x] = c_0 e_1 + c_1 e_2$, $c_0 \neq 0$, and $c_1^2 < -4c_0$.

If N is abelian, denote the algebras in this class by $L(c_0, c_1)$. Now $-4c_0 > c_1^2 \geqslant 0$, so $c_0 < 0$ and $(-c_0)^{1/2} \in F$. Putting $\alpha = (-1/c_0)^{1/2}$, we see from Theorem 2.3 that $L(c_0, c_1) \cong L(1, c_1')$. We must have $(c_1')^2 < 4$, so that $-2 < c_1' < 2$. Theorem 2.3 further shows that $L(1, c_1) \cong L(1, c_1')$ if and only if there is an element $\alpha \in F$ such that $\alpha c_1 = c_1'$ and $\alpha^2 = 1$; i.e., if and only if $c_1 = \pm c_1'$. Hence L is as described in (iv).

If N is not abelian, then $[e_1, e_2] \neq 0$; put $[e_1, e_2] = e_3$. Theorem 3.2(ii) implies that $c_1 = 0$. Furthermore, Corollary 4.2 gives that $\operatorname{ad} x|_{N^2} = 0$. Replacing x by $(-1/c_0)^{1/2}x$ and putting $g_1 = e_1$, $g_2 = (-1/c_0)^{1/2} e_2$, $g_3 = (-1/c_0)^{1/2} e_3$, we have the structure given in (v).

(b) Suppose now that L is not solvable. By Levi's theorem, L has a simple subalgebra and so must be simple. But any real simple Lie algebra contains a simple 3-dimensional subalgebra (see, for instance, the proofs of Theorems 2.4 and 2.5 and Lemma 2.2 in [2]). It follows that L must be 3-dimensional. Finally, L must be nonsplit, since otherwise it has a 2-dimensional nonabelian subalgebra. ∎

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